

## 6.1 The Basic Completion Algorithm

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# 6. Completion of Term Rewrite Systems

6.1. Basic Completion Algorithm

6.2. Improved Completion Algorithms

6.3. Using Completion to Perform Automated  
Induction Proofs

## 6.1. The Basic Completion Algorithm

Ex. 6.1.1  $\mathcal{E}$  : plus-rules

$\mathcal{R}$  : plus-rules oriented from left to right

$\mathcal{R}$  terminates (using LPO).

$\mathcal{R}$  is confluent ( $\mathcal{R}$  is even non-overlapping).

$\Rightarrow \mathcal{R}$  is convergent, thus it can directly be used as a decision procedure for the word problem.

Ex. 6.1.2  $\mathcal{E}$  : group axioms

$\mathcal{R}_0$  : group axioms oriented from left to right

$\mathcal{R}_0$  terminates (using RPOs where  $f$  has status  $\langle 2, 1 \rangle$  and precedence  $f \geq e$ )

For Confluence, we compute the 3 critical pairs of  $\mathcal{R}_0$ .

One of them is not joinable  $\Rightarrow \mathcal{R}_0$  not confluent.

Now: do not give up, but try to resolve the reason

that destroys confluence.

If there is a critical pair  $\langle s, t \rangle$  that cannot be joined, then compute the normal forms  $s'$  of  $s$  and  $t'$  of  $t$ , and add the rule  $s' \rightarrow t'$  or  $t' \rightarrow s'$ . Here, we check whether  $s' \succ t'$  or  $t' \succ s'$  holds for the relation  $\succ$  that we used to prove termination of  $\mathcal{R}$ . Here, it is advantageous if certain parameters of  $\succ$  (like status or precedence) are not yet completely fixed.

Ex. 6.12 (cont.) We now add the rule

$$f(f(x,y), i(y)) \rightarrow x \quad (64)$$

In this way, one automatically speculates lemmas that can be useful in further proofs.

Properties of  $\mathcal{R}_1$ :

- Is  $\mathcal{R}_1$  still equivalent to the original equations  $E$ ? (i.e.,  $\xrightarrow{*}_{\mathcal{R}_1} = \xrightarrow{*}_E$ )?

YES, because for the new rule  $s' \rightarrow t'$  we have

$$s' \xrightarrow{*}_{\mathcal{R}_0} t'$$

Thus:  $\xrightarrow{*}_{\mathcal{R}_1} = \xrightarrow{*}_{\mathcal{R}_0} = \xrightarrow{*}_E$ .

- Is  $\mathcal{R}_1$  terminating? YES, because  $s' > t'$  for the relation  $>$  that was already used in the termination proof of  $\mathcal{R}_0$ .
- Is  $\mathcal{R}_1$  confluent? The old critical pair can now be joined, but the new rule can create new critical pairs.  $\Rightarrow$  Confluence is not guaranteed and must be checked.

In our example,  $\mathcal{R}_1$  is not confluent.

(64) and (62) result in the following critical pair:

$$\begin{array}{ccc}
 f(f(x,e), i(e)) & & \\
 (G4) \swarrow & \searrow (G2) & \\
 x & f(x, i(e)) & \Rightarrow \text{add another} \\
 & & \text{rule} \\
 & & f(x, i(e)) \rightarrow x
 \end{array}$$

$\mathcal{R}_1$  creates 5 new critical pairs,  
4 of these 5 pairs are not joinable.

↑  $\mathcal{R}_2$  gets new rule (65)-(69).

$\mathcal{R}_2$  is still not confluent...

This completion process is repeated until no new rules are added anymore (i.e., until all critical pairs are joinable).

There are 3 possibilities:

1. After  $n$  iterations one reaches a TRS  $R_n$  whose critical pairs are joinable. Then  $R_n$  is convergent and equivalent to the original equations  $E$ . SUCCESS
2. One could reach a  $R_n$  whose termination FAIL cannot be proved. This could be because  $R_n$  is really not terminating or because our termination techniques are too weak. Here, we at least notice the failure.
3. The process might be non-terminating. All  $R_i$  are terminating and equivalent to  $E$ , but not confluent. NON-TERMINATION

Completion can't always succeed, because the word problem is undecidable. But one can try to improve the completion algorithm to make it succeed more often.

### Algorithm BASIC-COMPLETION

#### Ex 6.1.3 (Success of BASIC-COMPLETION)

Central Grupoids (slide)

For the critical pairs, unify non-variable subterms of  $f(\underline{f(x,y)}, \underline{f(y,z)})$  with

$$f(f(x, y), f(y, z)).$$

Here, all critical pairs of  $\mathcal{R}_1$  are joinable.

$\Rightarrow$  Alg. returns  $\mathcal{R}_n$ .

### Ex. 6.14. (Failure of BASIC-COMPLETION)

$$\mathcal{E} = \{ f(x, g(y, z)) \equiv g(f(x, y), f(x, z)), \\ f(g(x, y), z) \equiv g(f(x, z), f(y, z)) \}$$

$f$  is left- and right-distributive over  $g$

As  $\succ$ , we choose an RPO with  $f \sqsupseteq g$ .

$$\mathcal{R}_0 = \{ f(x, g(y, z)) \rightarrow g(f(x, y), f(x, z)), \quad (1) \\ f(g(x, y), z) \rightarrow g(f(x, z), f(y, z)) \} \quad (2)$$

$$f(g(x, y), g(y', z'))$$

$\swarrow^{(1)}$                              $\searrow^{(2)}$

$$g(f(g(x, y), y'), f(g(x, y), z')) \quad g(f(x, g(y, z')), f(y, g(y, z'))) \\ \downarrow * \qquad \qquad \qquad \downarrow *$$

$$g(g(f(x, y'), f(y, y')), g(f(x, z'), f(y, z'))) \quad g(g(f(x, y'), f(x, z')), g(f(y, y'), f(y, z')))$$

These terms cannot be turned into a terminating rule!

$\Rightarrow$  Fail.

### Ex 6.15. (Non-Termination of BASIC-COMPLETION)

$$\mathcal{E} = \{ f(g(f(x))) \equiv g(f(x)) \}.$$

One can remove the outer  $f$  if there is one  $g$  between two  $f$ 's.

$$\mathcal{R}_0 = \{ f(g(f(x))) \rightarrow g(f(x)) \}.$$

$$f(g(f(g(f(x))))) \quad \begin{matrix} \swarrow \\ g(f(g(f(x)))) \end{matrix} \quad \begin{matrix} \searrow \\ f(g(g(f(x)))) \end{matrix}$$

$\downarrow$

$$g(g(f(x)))$$

$$\mathcal{R}_1 = \{ f(g(f(x))) \rightarrow g(f(x)), \\ f(g^2(f(x))) \rightarrow g^2(f(x)) \}$$

One can remove the outer  $f$  if there are one or two  $g$ 's between the  $f$ 's.

$$f g^2 f g f(x) \quad \begin{matrix} \swarrow \\ f f g f(x) \end{matrix} \quad \begin{matrix} \searrow \\ f g^3 f(x) \end{matrix} \quad f g f g^2 f(x) \quad \begin{matrix} \swarrow \\ g f g^2 f(x) \end{matrix} \quad \begin{matrix} \searrow \\ f g^3 f(x) \end{matrix} \quad f g^2 f g^2 f(x) \quad \begin{matrix} \swarrow \\ g^2 f g^2 f(x) \end{matrix} \quad \begin{matrix} \searrow \\ f g^4 f(x) \end{matrix}$$

$\downarrow$

$$g^3 f(x) \quad g^3 f(x) \quad g^4 f(x)$$

$$\mathcal{R}_2 = \{ f(g^i(f(x))) \rightarrow g^i(f(x)) \mid 1 \leq i \leq 4 \}$$

This can be repeated infinitely many times, which yields

$$\mathcal{R}_\infty = \{ f(g^i(f(x))) \rightarrow g^i(f(x)) \mid 1 \leq i \}.$$

$\Rightarrow$  BASIC-COMPLETION does not terminate.

If BASIC-COMPLETION does not terminate, then it generates an infinite TRS  $\mathcal{R}_\infty$  that is convergent and equivalent to  $\mathcal{E}$ .

Thus: then BASIC-COMPLETION can still be used as a semi-decision procedure for the word problem:

To check  $s \equiv_{\mathcal{E}} t$ :

- Compute  $\mathcal{R}_0$  and check whether  $s \downarrow_{\mathcal{R}_0} t$ .

If yes, stop and return "true".

If no:

- Compute  $\mathcal{R}_1$  and check whether  $s \downarrow_{\mathcal{R}_1} t$ .

If yes, stop and return "true".

If no:

- Compute  $\mathcal{R}_2 \dots$

If  $s \equiv_{\mathcal{E}} t$ , then  $s \downarrow_{\mathcal{R}_{\infty}} t$ . Thus, there exists an  $n \in \mathbb{N}$  with  $s \downarrow_{\mathcal{R}_n} t$ . Therefore, the algorithm returns "true" after  $n$  steps.

### Thm 6.1.6. (Correctness of BASIC-COMPLETION)

Let  $\mathcal{E}$  be a (finite) set of equations, let  $\succ$  be a reduction order.

(a) If  $\text{BASIC-COMPLETION}(\mathcal{E}, \succ)$  terminates with result  $\mathcal{R}_n$ , then  $\mathcal{R}_n$  is a finite convergent TNS that is equivalent to  $\mathcal{E}$ .

Thus,  $\mathcal{R}_n$  can be used to obtain a decision procedure for the word problem of  $\mathcal{E}$  (by the alg. WORD-PROBLEM).

(b) If  $\text{BASIC-COMPLETION}(\mathcal{E}, \succ)$  does not terminate, then  $\mathcal{R}_{\infty} = \bigcup_{i \geq 0} \mathcal{R}_i$  is an infinite convergent TNS that is equivalent to  $\mathcal{E}$ . Thus, BASIC-COMPLETION can be used as a

Semi-decision procedure for the word problem of  $E$ .

Proof: (a) •  $\mathcal{R}_n$  is finite because each step of B-C only adds finitely many rules.

- $\mathcal{R}_n$  terminates since  $l > r$  for all  $l \rightarrow r \in \mathcal{R}_n$
- $\mathcal{R}_n$  is confluent, because all  $\langle s, t \rangle \in CP(\mathcal{R}_n)$  are joinable.
- $\mathcal{R}_n$  is equivalent to  $E$ :
  - $\mathcal{R}_0$  is adequate for  $E$  &  $\mathcal{R}_n \supseteq \mathcal{R}_0$  is also adequate for  $E$
  - $\mathcal{R}_n$  is correct for  $E$  since only rules from critical pairs were added during completion.

(b) •  $\mathcal{R}_\infty$  is infinite because each iteration adds at least one rule.

- $\mathcal{R}_\infty$  terminates as in (a).
- $\mathcal{R}_\infty$  is confluent:

If  $l_1 \rightarrow r_1, l_2 \rightarrow r_2 \in \mathcal{R}_\infty$  form a critical pair,  
then there is an  $n \in \mathbb{N}$  with  $l_1 \rightarrow r_n, l_2 \rightarrow r_n \in \mathcal{R}_n$ .

Then this critical pair is joinable in  $\mathcal{R}_{n+1}$   
(and thus, also in  $\mathcal{R}_\infty \supseteq \mathcal{R}_{n+1}$ ).

- $\mathcal{R}_\infty$  is equivalent to  $E$  as in (a). □